Fractional evolution system modelling Mixing in milli torus reactor

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ABSTRACT

In this work, We study the local existence of mild solutions for a class of fractional evolution system for higher-order semilinear parabolic equations with nonlocal terms. We provide a fractional model for the dynamic process of tracer response in aerated and non-aerated conditions, using the variable t (time) in considered systems of partial fractional differential equations is considered in the Caputo sense, and the variable x (space) in the Riemann-Liouville sense.
Keywords: Reaction-diffusion systems, mild solution, fractional integrals, Caputo derivatives, Putzer algorithm, milli torus reactor, mixing.

1 INTRODUCTION

Mixing plays a very important role in different industries such as chemistry, biochemistry, food,.., and the The flow behaviour inside the milli torus reactor was modelled classically by the dispersed plug flow model. Correlations have been proposed to predict gas hold up and axial dispersion [6]. These can be treated by a methods involving fractional calculus [2], [8].

IN this work, we study a class of semilinear parabolic system involving fractional derivatives and models

$$\begin{cases} {}^cD_t^{\alpha_1}u + (-\Delta)^{m_1}u = J_{0|t}^{\beta_1}(|v|^{p-1}v(t)) \quad, \\ {}^cD_t^{\alpha_2}v + (-\Delta)^{m_2}v = J_{0|t}^{\beta_2}(|u|^{q-1}u(t)) \quad, \\ u(x,0) = u_0(x), \quad v(0,x) = v_0(x) \end{cases} \quad (1)$$

where $(x,t) \in R^N \times R_+$, $p,q > 1$, $m_1, m_2 \in R^*$, $\alpha_1, \alpha_2 \in (0,1)$ with $m_1 \neq m_2$, $\alpha_1 \neq \alpha_2$ and $\beta_1 \neq \beta_2$. We denote by ${}^cD_t^{\alpha_i} \ \forall i = 1,2$ the Caputo derivative and $J_{0|t}^{\alpha_i} \ \forall i = 1,2$ the Riemann-Liouville fractional integrals. We also, denote by $(-\Delta)^{m_i}$ the infinitesimal generator of a strongly continuous semigroup $e^{(-\Delta)^{m_i}t}$ that is not order-preserving.

The $m-$laplacian $(-\Delta)^m$ has for eigenvalues [3]

$$\lambda_n = \left(\frac{n\pi}{2} - \frac{(1-m)\pi}{4} \right)^{2m} + o\left(\frac{1}{n} \right) \quad (2)$$

The plan of this paper is as follows. In Section 2, we state some definitions and results needed. section 3 is devoted to the local existence and uniqueness of mild solutions for $\alpha_1 = \alpha_2 = 0$ in the whole space, while subsection 4 is on a direct resolution of the homogeneous system (1) in the interval. Finally we give physical interpretation and conclusion.

2 PRELIMINARIES

Before proving our main results, we need to recall some basic definitions and properties.

2.1 Basic definitions (see [2], [8])

Let f be an integrable function on R^+, then for $0 < T < \infty$ and $t \in [0,T]$, we call

$$J_{0|t}^{\alpha}f(t): \quad = \quad \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}}d\tau, \quad t > 0, \quad (3)$$

$$J_{t|T}^{\alpha}f(t): \quad = \quad \frac{1}{\Gamma(\alpha)} \int_t^T \frac{f(\tau)}{(\tau-t)^{1-\alpha}}d\tau, \quad t > 0, \quad (4)$$

$$J_{0|t}^{0}f(t): \quad = \quad f(t)$$

the left-handed Riemann-Liouville fractional integral and the right-handed Riemann-Liouville fractional integral of order $\alpha \in]0,1[$, where $\Gamma(\alpha)$ is the Euler Gamma function.

The left-handed and right-handed Caputo derivatives

$^cD^\alpha_{0|t}$ and $^cD^\alpha_{t|T}$ of order $\alpha \in\,]0,1[$ are defined by

$$
\begin{aligned}
^cD^\alpha_{0|t}f(t) : &= (J^{n-\alpha}_{0|t}D^n)f(t) \\
&= \frac{1}{\Gamma(n-\alpha)}\frac{d}{dt}\int_0^t \frac{f^n(\tau)}{(t-\tau)^{\alpha-n+1}}\,d\tau, \quad (5)
\end{aligned}
$$

$$
\begin{aligned}
^cD^\alpha_{t|T}f(t) : &= (-1)^n(J^{n-\alpha}_{t|T}D^n)f(t) \\
&= \frac{(-1)^n}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^T \frac{f^n(\tau)}{(\tau-t)^{\alpha-n+1}}\,d\tau, \quad (6)
\end{aligned}
$$

hence, the following properties:

$$
\begin{aligned}
D^{1+\alpha}_{t|T}f : &= -D.D^\alpha_{t|T}f, && (7) \\
D^\alpha_{0|t}J^\alpha_{0|t} : &= Id_{L^q(0,T)} \quad \text{for all} \quad 1\le q \le \infty, && (8) \\
D^0_{0|t}f(t) : &= f(t), && (9)
\end{aligned}
$$

hold for all $t \in [0,T]$, where $T > 0$.
We denote by $S(t) = e^{-t(-\Delta)^m}$ the strongly continuous semigroup on $L^2(R^N)$ generated by $(-\Delta)^m$.

3 LOCAL EXISTENCE AND UNIQUENESS

We shall establish the existence of a unique local mild solution by using properties of semigroup

3.1 Local existence

Before giving local existence, we state the definition of a mild solution of (1) for $\alpha_1 = \alpha_2 = 0$.
Definition 1. Let $u_0, v_0 \in C_0(R^N)$. Let $p, q > 1$, $\beta_i \in (0,1)$ and $m_i \ge 1$ for $i = 1,2$.
A couple $(u,v) \in C([0,T], C_0(R^N) \times C_0(R^N))$ is said to be a mild solution of the problem (1) if the following integral equations hold for every $T > 0$

$$
u(t) = S_1(t)u_0 + \int_0^t S_1(t-s)J^{\beta_1}_{0|s}(|v|^{p-1}v)ds, \quad t \in [0,T]
$$

$$
v(t) = S_2(t)v_0 + \int_0^t S_2(t-s)J^{\beta_2}_{0|s}(|u|^{q-1}u)ds, \quad t \in [0,T],
$$

where $S_i(t) := e^{(-\Delta)^{m_i}t}$ for $i = 1,2$ is the strongly continuous semigroup on $L^2(R^N)$ generated by the Laplacian operator $(-\Delta)^{m_i}$.

Theorem 1.
Assume $p, q > 1$, $\beta_i := 1 - \gamma_i \in (0,1)$ and $m_i \in N^*$ $(i = 1,2)$. For each initial data
$u_0, v_0 \in C_0(R^N)$ and $T = T(u_0,v_0) > 0$ such that
$\|u_0\|_\infty, \|v_0\|_\infty \le \frac{A_0}{2}$ for $A_0 > 0$ and
$max\{2C(p,q),1\}\tilde{T}(u_0,v_0) \le \frac{1}{2}$, there exists a unique mild solution
$(u,v) \in \{C([0,T_{max}), C_0(R^N))\}^2$ defined on $(0,T_{max})$
satisfying the alternative:
Either $T_{max} = +\infty$

or $T_{max} < +\infty$
and $\lim_{t \to T_{max}} (\|u(t)\|_{L^\infty(R^N)} + \|v(t)\|_{L^\infty(R^N)}) = \infty$.
where
$T_{max} := \sup\{T > 0 : u \text{ is a mild solution to }(1)\}$
in $L^\infty((0,T), C_0(R^N) \times C_0(R^N))[7]$.

Proof. • By hypothesis $\|u_0\|_\infty, \|v_0\|_\infty \le \frac{A_0}{2}$ for $A_0 > 0$. We consider the set

$$
E_T = \left\{(u,v) \in L^\infty((0,T), \{C_0(R^N)\}^2); \ \||(u,v)\|| \le 2A_0\right\},
$$

where $\||(u,v)\|| = \|u_0\|_\infty + \|v_0\|_\infty$, $T > 0$, for which will be precised later and $\||\cdot\||$ is the norm of E_T defined by

$$
\||(u,v)\|| = \|u\|_1 + \|v\|_1 = \|u\|_{L^\infty((0,T)\times R^N)} + \|v\|_{L^\infty((0,T)\times R^N)}.
$$

In what follows, we prove that u, v are well defined and map E_T into itself, and are a contraction mapping from E_T into itself provided T is sufficiently small. For that, we introduce the following notations

$$
\|\cdot\|_\infty := \|\cdot\|_{L^\infty(R^N)}, \quad \|\cdot\|_{\infty,T} := \|\cdot\|_{L^\infty(0,T)}
$$

Consider the mapping $\Psi(u,v) = (\Psi_1(u,v), \Psi_2(u,v)) : E_T \to L^\infty\left((0,T), \{C_0(R^N)\}^2\right)$ defined by

$$
\begin{aligned}
\Psi_1((u,v)) = \ & S_1(t)u_0 + \frac{1}{\Gamma(1-\gamma_1)}\int_0^t S_1(t-s) \\
& \int_0^s (s-\sigma)^{-\gamma_1}|v|^{p-1}v(\sigma)\,d\sigma\,ds \quad (10)
\end{aligned}
$$

$$
\begin{aligned}
\Psi_2((u,v)) = \ & S_2(t)v_0 + \frac{1}{\Gamma(1-\gamma_2)}\int_0^t S_2(t-s) \\
& \int_0^s (s-\sigma)^{-\gamma_2}|u|^{q-1}u(\sigma)\,d\sigma\,ds \quad (11)
\end{aligned}
$$

For $(u,v) \in E_T$ and according to the semigroup properties, we have

$$
\begin{aligned}
\||\Psi(u,v)\|| \le\ & (\|u_0\|_\infty + \|v_0\|_\infty) \\
& + \left\|\int_0^t\int_\sigma^t \frac{(s-\sigma)^{-\gamma_1}}{\Gamma(1-\gamma_1)}\|v(\sigma)\|_\infty^p\,ds\,d\sigma\right\|_{\infty,T} \\
& + \left\|\int_0^t\int_\sigma^t \frac{(s-\sigma)^{-\gamma_2}}{\Gamma(1-\gamma_2)}\|u(\sigma)\|_\infty^q\,ds\,d\sigma\right\|_{\infty,T} \\
\le\ & A_0 + 2T(u_0,v_0)A_0, \quad (12)
\end{aligned}
$$

where

$$
T(u_0,v_0) = max\left\{\frac{T^{2-\gamma_1}2^{p-1}A_0^{p-1}}{\Gamma(3-\gamma_1)}, \frac{T^{2-\gamma_2}2^{q-1}A_0^{q-1}}{\Gamma(3-\gamma_2)}\right\}
$$

If $2T(u_0,v_0) \le 1$, then $\||\Psi(u,v)\|| \le 2A_0$ and $\Psi(u,v) \in E_T$.

• It remains to prove that Ψ is a contraction. For that, let $(u_1, v_1), (u_2, v_2) \in E_T$, using the following estimate

$$\|u^p - v^p\|_\infty \leq C(p)\|u - v\|_\infty \left(\|u\|_\infty^{p-1} + \|v\|_\infty^{p-1}\right), \quad (13)$$

for all u, v and $p \geq 1$, we have

$$\||\Psi(u_1, v_1) - \Psi(u_2, v_2)\||$$

$$\leq \left\| \int_0^t \int_0^s \frac{(s-\sigma)^{-\gamma_1}}{\Gamma(1-\gamma_1)} \|\mathcal{Y}_p\|_\infty \, d\sigma \, ds \right\|_{\infty,T}$$

$$+ \left\| \int_0^t \int_0^s \frac{(s-\sigma)^{-\gamma_2}}{\Gamma(1-\gamma_2)} \|\mathcal{Y}_p\|_\infty \, d\sigma \, ds \right\|_{\infty,T}$$

$$\leq 2C(p,q)\,T(u_0, v_0)\||(u_1, v_1) - (u_2, v_2)\||$$

$$\leq \frac{1}{2}\||(u_1, v_1) - (u_2, v_2)\||,$$

with,

$$\mathcal{Y}_p := |v_1|^{p-1} v_1(\sigma) - |v_2|^{p-1} v_2(\sigma)$$

and,

$$\mathcal{Y}_q := |u_1|^{q-1} u_1(\sigma) - |u_2|^{q-1} u_2(\sigma)$$

Thus Ψ is a strict contraction in E_T provided that T satisfy the following condition:

$$max\,\{2C(p,q), 1\}\,T(u_0, v_0) \leq \frac{1}{2}$$

where $C(p,q) = max\,\{C(p), C(q)\}$. We conclude by the Banach fixed point theorem that $(u, v) \in E_T$ is a mild solution of the problem (1).

The uniqueness of the solution of the problem (1) holds by the singular Gronwall's lemma. Hence, by the Banach fixed point theorem, the system admits a unique mild solution.

3.2 Resolution of Homogeneous system

We are interesting on a direct resolution of system (1) in $(-1, 1) \times R_+$ without second member, which corresponds to the system

$$\begin{cases} {}^cD_t^{\alpha_1} u + (-\Delta)^{m_1} u = 0 \\ {}^cD_t^{\alpha_2} v + (-\Delta)^{m_2} v = 0 \\ u(x, 0) = u_0(x), \quad v(0, x) = v_0(x) \end{cases}, \quad (14)$$

The system (14) can be written as follows

$$\begin{pmatrix} u^{(\alpha)} \\ v^{(\alpha)} \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix} \quad (15)$$

where, $u^{(\alpha)}$ and $v^{(\alpha)}$ stands for the caputo derivative, and A is the 2×2 matrix defined by

$$\begin{pmatrix} (-\Delta)^{m_1} & 0 \\ 0 & (-\Delta)^{m_2} \end{pmatrix}$$

Theorem 2.
Let λ_1, λ_2 two distinct eigenvalues of matrix A.

For $u_0 > 0$, $v_0 > 0$, the system (14) has the explicit solutions

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} e^{\lambda_1^{1/\alpha} t} M_0 + \dfrac{\left(e^{\lambda_1^{1/\alpha} t} - e^{\lambda_2^{1/\alpha} t}\right)}{\lambda_1^{1/\alpha} - \lambda_2^{1/\alpha}} M_1 \end{pmatrix}$$

$$\times \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \quad (16)$$

For the proof, we need the following results
Theorem (Putzer Algorithm for finding e^{At})[5]
Let $\lambda_1, \lambda_2, \cdots \lambda_N$ be the (not necessarily distinct) eigenvalues of the matrix A. Then

$$e^{At} = \sum_{k=0}^{N-1} p_{k+1}(t) M_k, \quad (17)$$

where $M_0 := I$,

$$M_k := \prod_{i=1}^{k} (A - \lambda_i I), \quad (18)$$

for $1 \leq k \leq N$ and the vector function p defined by

$$p(t) = \begin{bmatrix} p_1(t) \\ p_2(t) \\ \cdots \\ p_N(t) \end{bmatrix},$$

for $t \in R$, is the solution of the IVP

$$p' = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 1 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 1 & \lambda_3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & \lambda_N \end{bmatrix} p, \quad p(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Corollary 1 (see [1]).
Let λ be a parameter, the Mittag Leffler functions can be written explicitly in terms of exponential functions as follow

$$E_\alpha(\lambda \frac{d}{dx}) = e^{\lambda^{\frac{1}{\alpha}} \frac{d^{\frac{1}{\alpha}}}{dx^{\frac{1}{\alpha}}}}$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} e^{s\xi} \mathcal{A}_N^{(\beta)}(\xi, \lambda^{\frac{1}{N\alpha}}) \, d\xi, .$$

where $\frac{1}{\alpha} = n\beta, \; 0 < \beta \leq 1$,
and the function $\mathcal{A}_N^{(\alpha)}(\xi, \lambda)$ is given by

$$\mathcal{A}_N^{(\alpha)}(\xi, \lambda) = -\frac{1}{\xi} \int_{-\infty}^{\infty} \mathcal{A}_N(\tau, \lambda) W(-\alpha, 0; \tau(-\xi)^{-\alpha}) \, d\tau.$$

where the functions $\mathcal{A}_N(\tau, \lambda)$ and $W(-\alpha, 0; -\lambda^\alpha t^{-\alpha})$ are represented by

$$\mathcal{A}_N(\tau, \lambda) = \begin{cases} \int_0^\infty \cos(r\xi + (-1)^{\frac{N+1}{2}} \lambda^N r^N) \, dr & N = 2k+1 \\ \frac{1}{N\lambda} \int_0^\infty e^{-r} \frac{\cos(\frac{1}{\lambda} r^{\frac{1}{N}} \xi)}{r^{1-\frac{1}{N}}} \, dr & N = 4k+2 \\ \frac{1}{N\lambda} \int_0^\infty e^r \frac{\cos(\frac{1}{\lambda} r^{\frac{1}{N}} \xi)}{r^{1-\frac{1}{N}}} \, dr & N = 4k \end{cases}$$

and,

$$\frac{1}{t} W(-\alpha, 0; -\lambda^\alpha t^{-\alpha}) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{-\lambda^\alpha s^{-\alpha}} e^{st} \, ds$$

Proof of Theorem 2.

The proof is carried out in very much the same way as the direct proof of Theorem 3.2. in [1]. By writing solutions of (14) in terms of elementary function (exponential functions), we use Putzer algorithm for writing $e^{A\lambda}$.

Let λ_1, λ_2 be the two distinct eigenvalues of the matrix A. Then we can write $e^{A\lambda}$ as given in (17), and for $k = 1, 2$ the functions p_k are given by the relations

$$p_1(\lambda) = e^{\lambda_1 \lambda} \tag{19}$$

$$p_i(\lambda) = e^{\lambda_i \lambda} \int_0^t e^{-\lambda_i \tau} p_{i-1}(\tau) \, d\tau, \quad i = 1, 2. \tag{20}$$

Applying the Laplace transform on both sides of equations of system (15), we get

$$\begin{pmatrix} u \\ v \end{pmatrix} = \mathcal{L}^{-1} \left\{ \frac{s^{\alpha-1}}{s^\alpha I - A} \right\} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \tag{21}$$

$$= E_\alpha(At^\alpha) \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \tag{22}$$

The Putzer algorithm for an exponential matrix e^{tA}, and corollary 1, give

$$\begin{pmatrix} u \\ v \end{pmatrix} = e^{A \frac{1}{\alpha} t} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = (p_1(t)M_0 + p_2(t)M_1) \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \tag{23}$$

where for the eigenvalues λ_1, λ_2 of A, we have

$$p_1(t) = e^{\lambda_1 t}, \quad p_2(t) = \frac{1}{\lambda_1 - \lambda_2} \left(e^{\lambda_1 t} - e^{\lambda_2 t} \right). \tag{24}$$

Then, from (23) and (24), we get (16).

4 PHYSICAL INTERPRETATIONS AND CONCLUSIONS

A physical meaning of (14) by taking u, v : as tracer concentrations at different times in the work of [6], cited above. So, the residence time distribution (RTD) was (classically) an important parameter characterizing the flow pattern of the mixing and the flow pattern occurring inside the reactor, but only axial dispersion were considered, and the radial one were neglected. However, the presence of term $(-\Delta)^m$, $m \in N^*$ [4] can compensate this, in terms that the corresponding evolution processes are not order-preserving, which motivated as to construct for given model solutions (physically concentrations) that may be exits all the time by resolving directly the system (14) in many phases in the torus reactor even to have concentrations with no lack. Thus fractional model is a new and power model for studying evolution processes at different times without absence of memory.

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